Tim Cohen PHYS 741 F24

Lecture 3 - Spontaneous Symmetry Breaking

Goal: describe theories with symmetry that is hidden because vacuum (ground state solution) spontaneously breaks the symmetry

Two important implications:

1) Massless Goldstone bosons appear for each broken symmetry They have interactions dictated by symmetry breaking pattern

2) If symmetry is local (gauged) $\Rightarrow$ Some gauge bosons eat the would be Goldstones and become massive (Higgs mechanism)

Spontaneous breaking of Discrete Symmetry

φ is real scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

Has symmetry $Z^{-1}\varphi(x)Z = -\varphi(x)$ is so-called $\mathbb{Z}_2$ symmetry

s.t. $Z^2 = 1 \Rightarrow Z = Z^{-1}$ and uniterity $\Rightarrow Z = Z^\dagger \Rightarrow Z^{-1} = Z^\dagger$

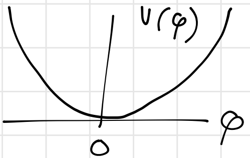
Check for min of energy: (do pert theory about vacuum state w/ min E)

$$H = \int d^3x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4\right]$$

"Vacuum expectation value (vev)"

See H is minimized for uniform field $\varphi(x) = v \Rightarrow \Pi(x) = \vec{\nabla}\varphi(x) = 0$

$\Rightarrow$ Minimize $V(\varphi) \Rightarrow$



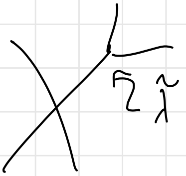
$\Rightarrow v = 0 \Rightarrow$ unbroken symmetry

Note the $\mathbb{Z}_2$ symmetry forbids a $\frac{\tilde{\lambda}}{3!}\varphi^3$ term in $\mathcal{L}$

⇒ This theory (w/ $\tilde{\lambda}=0$) predicts scattering of states w/ mass μ only involving an even number of particles: $\mathcal{M}(\varphi\varphi \to \varphi\varphi\varphi) = 0$

If $\mathbb{Z}_2$ were broken then $\mathcal{M}(\varphi\varphi \to \varphi\varphi\varphi) \neq 0$.

eg



If $\tilde{\lambda}$ appears in $\mathcal{L}$, call this explicit breaking. (When $\tilde{\lambda}$ small, treat as a "spurion" of symmetry breaking.)

In this case λ & $\tilde{\lambda}$ are independent parameters.

When symmetry is "spontaneously broken" we will see that $\tilde{\lambda}$ and λ are related (the broken theory has the same number of params as the original theory.)

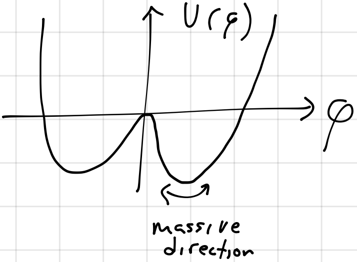
3.3

To break the symmetry, note that nothing stopping us from writing $m^2 \to -\mu^2 \Rightarrow$

← mass squared parameter, no longer mass of particle

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}\mu^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

Want to minimize energy $\Rightarrow \varphi(x) = v \Rightarrow$ minimize potential:



$$U(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4!}\varphi^4$$

$$\left.\frac{dU}{d\varphi}\right|_{\varphi=v} = 0 = -\mu^2\varphi + \frac{\lambda}{6}\varphi^3\Big|_{\varphi=v}$$

$\Rightarrow$ two sols $\Rightarrow v = \pm\sqrt{\frac{6}{\lambda}}\mu$

Check units: $[\varphi] = [v] = 1$, $[\mu] = 1$, $[\lambda] = 0$ ✓

The two sols $\Rightarrow$ two vacuua $|\Omega_+\rangle$ and $|\Omega_-\rangle$ s.t.

$$\langle\Omega_\pm|\varphi(x)|\Omega_\pm\rangle = \pm v \quad \text{and} \quad \langle\Omega_+|\Omega_-\rangle = 0$$

In principle, one can tunnel from $|\Omega_\pm\rangle \to |\Omega_\mp\rangle$ but would require all creation operators to tunnel together $\Rightarrow$ exponentially suppressed by e^{-V} w/ V is volume of space-time

$\Rightarrow$ Treat $|\Omega_+\rangle$ and $|\Omega_-\rangle$ as independent disconnected vacuua

Want to pick a vacuum and do perturbation theory

Choose $|\Omega_+\rangle$ and define $\sigma(x) = \varphi(x) - v$

$$\Rightarrow \langle\Omega_+|\sigma(x)|\Omega_+\rangle = 0$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4$$

Note: linear term for $\sigma(x)$ canceled as it must since this is exitation about the minimum (show this yourself!)

Mass of σ: $\frac{\partial^2 V}{\partial \sigma^2} = 2\mu^2 \Rightarrow M_\sigma = \sqrt{2}\mu$

Simplest example of spontaneously broken symmetry.

No longer manifests Z_2 explicitly, <u>but</u> Z_2 reflected by relation between mass and interaction terms in $\mathcal{L}$

Spontaneous breaking of continuous symmetry

Work with "linear sigma model"

N real scalar fields $\varphi^i(x)$ with (repeated i's are summed)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi^i)^2 + \frac{1}{2}\mu^2(\varphi^i)^2 - \frac{\lambda}{4}[(\varphi^i)^2]^2$$

(change norm of λ term for convenience)

$\mathcal{L}$ is invariant under $\varphi^i \to O^{ij}\varphi^j$ for any $N\times N$ orthogonal matrix O (s.t. $O^T = O^{-1}$) This is just rotation group in N-dim a.k.a the "special orthogonal group" $SO(N)$

There is no $SO(1)$. $SO(2)$ is rotations in the plane $\Rightarrow$ 1 param

$SO(3)$ is 3-d rotations $\Rightarrow$ 3 params

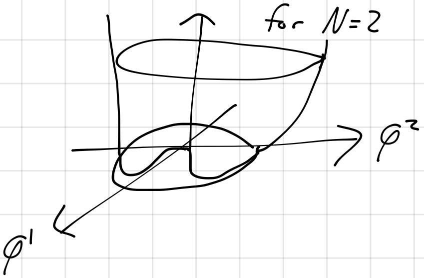
Can show in general $SO(N)$ has $N(N-1)/2$ continuous symmetries

3.5

Minimize

$$V(\varphi^i) = -\frac{1}{2}\mu^2(\varphi^i)^2 + \frac{\lambda}{4}\left[(\varphi^i)^2\right]^2$$

$$\Rightarrow (\varphi_0^i)^2 = \frac{\mu^2}{\lambda}$$



Flat direction around the trough. Massive direction up the sides

All vacuum choices will yield same physics $\Rightarrow$ Choose $\varphi_0^i = (0, \ ,0, v)$

with $v = \mu/\sqrt{\lambda}$

Then define shifted fields in new vacuum

$\varphi^i(x) = (\pi^k(x),\ v + \varphi(x))$ w/ $k = 1, \dots, N-1$ (dropped const terms)

Then
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3$$
$$-\sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}\left[(\pi^k)^2\right]^2$$

$\Rightarrow m_\sigma = \sqrt{2}\mu$ and $m_{\pi^k} = 0$

SO(N) symmetry is hidden SO(N-1) symmetry rotating π^k fields remains unbroken.

σ is in "radial" direction + π^k are in "tangential" directions

Goldstone's Theorem

Number of massless Goldstone bosons equals number of broken symmetries

This is true both classically and quantum mechanically (i.e. at loop level)

Classical proof:

Consider theory with several fields $\varphi^a(x)$ and Lagrangian

$$\mathcal{L} = (\text{terms w/ } \partial_\mu) - V(\varphi)$$

Let φ_0^a be constant that minimizes V: $\frac{\partial}{\partial\varphi^a} V \Big|_{\varphi^a(x) = \varphi_0^a} = 0$

Then expanding $V(\varphi)$ about its minimum yields

$$V(\varphi) = V(\varphi_0) + \frac{1}{2}(\varphi - \varphi_0)^a (\varphi - \varphi_0)^b \left(\frac{\partial^2}{\partial\varphi^a \partial\varphi^b} V\right)\Big|_{\varphi_0} + \dots$$

$\Rightarrow$ Coeff of quadratic term (gives mass matrix)

$\left(\frac{\partial^2 V}{\partial\varphi^a \partial\varphi^b}\right)\Big|_{\varphi_0} = m^2_{ab}$ is symmetric matrix whose eigenvalues give masses of particles

Want to show that every continuous symmetry of $\mathcal{L}$ that is <u>not</u> a sym of φ_0 $\Rightarrow$ zero eigenvalue

In general, sym transformation has form $\varphi^a \to \varphi^a + \alpha \Delta^a(\varphi)$

w/ α is infinitesmal sym param and Δ^a is function of φ^a's

Specalize to constant fields $\Rightarrow$ ∂_μ terms vanish

Invariance of $\mathcal{L} \Rightarrow V(\varphi^a) = V(\varphi^a + \alpha \Delta^a(\varphi)) \Leftrightarrow \Delta^a(\varphi)\frac{\partial}{\partial \varphi^a} V(\varphi) = 0$

Then differentiate wrt φ^b and set $\varphi = \varphi_0$

$$0 = \left(\frac{\partial \Delta^a}{\partial \varphi^b}\right)\Big|_{\varphi_0} \underbrace{\left(\frac{\partial V}{\partial \varphi^a}\right)\Big|_{\varphi_0}}_{\substack{\text{Vanishes since} \\ \varphi_0 \text{ is min}}} + \Delta^a(\varphi_0)\left(\frac{\partial^2}{\partial \varphi^a \partial \varphi^b} V\right)\Big|_{\varphi_0}$$

$$\Rightarrow \Delta^a(\varphi_0)\left(\frac{\partial^2}{\partial \varphi^a \partial \varphi^b} V\right)\Big|_{\varphi_0} = 0 \Rightarrow \begin{cases} \text{If trans leaves } \varphi_0 \text{ invariant} \Rightarrow \Delta^a(\varphi_0) = 0 \\ \text{If trans does } \underline{\text{not}} \text{ leave } \varphi_0 \text{ unchanged} \\ \Rightarrow \Delta^a(\varphi_0) \text{ is eigenvector of} \\ \left.\frac{\partial^2 V}{\partial \varphi^a \partial \varphi^b}\right|_{\varphi^0} \text{ w/ zero eigenvalue} \end{cases}$$

Another derivation of Goldstone's Thm

Continuous symmetry $\Rightarrow$ Noether current J_μ w/ $\partial_\mu J_\mu = 0$ (on EOM)

$\Rightarrow$ Conserved charge operator $Q = \int d^3x \, J_0(x) = \int d^3x \sum_m \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_m} \frac{\delta \varphi_m}{\delta \alpha}$

Recall, canonically conjugate fields $\Pi_m = \frac{\delta \mathcal{L}}{\delta \dot{\varphi}_m}$

w/ $[\varphi_n(\vec{x}), \Pi_m(\vec{y})] = i\,\delta^3(\vec{x} - \vec{y})\,\delta_{nm}$

$$\Rightarrow [Q, \varphi_n(\vec{y})] = \sum_m \int d^3x \left[\Pi_m(\vec{x}), \varphi_n(\vec{y})\right] \frac{\delta \varphi_m(\vec{x})}{\delta \alpha} = -i\,\frac{\delta \varphi_n(\vec{y})}{\delta \alpha}$$

$\Rightarrow Q$ generates the symmetry transformation

Conservation of charge $\Rightarrow [H, Q] = i\partial_t Q = 0$

Q corresponds to conserved charge no matter what vacuum we expand around

Spontaneous symmetry breaking occurs (by definition) if $Q|\Omega\rangle_{sym}$ is unstable and true (stable) vac carries charge

$Q|\Omega\rangle \neq 0$ If broken vacuum has energy E_0 $H|\Omega\rangle = E_0|\Omega\rangle$

$$\Rightarrow HQ|\Omega\rangle = [H, Q]|\Omega\rangle + QH|\Omega\rangle = E_0 Q|\Omega\rangle$$

$\Rightarrow$ the state $Q|\Omega\rangle$ has same energy as the ground state

Next, we can construct a state from the vacuum

$$|\pi(\vec{p})\rangle = \frac{-2i}{F}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}} J_0(x)|\Omega\rangle \quad \text{w/ energy } E_\pi = E(\vec{p}) + E_0$$

F is constant w/ $[F] = 1$ to get units correct & $-2i$ is for convenience

Then note $|\pi(\vec{0})\rangle = \frac{-2i}{F} Q|\Omega\rangle$ has energy E_0

$\Rightarrow E(\vec{p}) \to 0$ as $\vec{p} \to 0 \Rightarrow |\pi\rangle$ must have zero mass

These are the Goldstone bosons:

$$\langle\pi(\vec{q})|J_0(y)|\Omega\rangle = iq_0 F e^{i\vec{q}\cdot\vec{y}} \overset{\text{make Lorentz covariant}}{\Longrightarrow} \langle\Omega|J^\mu(x)|\pi(q)\rangle = iq^\mu F e^{-ipx}$$

Linear vs non-linear realization of GB

Specialize to complex scalar field theory w/ $U(1) = SO(2)$ symmetry

$$\mathcal{L} = |\partial_\mu \varphi|^2 + \mu^2 |\varphi|^2 - \frac{\lambda}{4} |\varphi|^4$$

has global sym: $\varphi \to e^{i\alpha} \varphi$ when $\langle \varphi \rangle = 0$

Minimize $V(\varphi) \Rightarrow |\varphi_0|^2 = \frac{2\mu^2}{\lambda}$

$\Rightarrow \infty$ family of equavelent vacuna w/ $\langle \Omega_\theta | \varphi | \Omega_\theta \rangle = \sqrt{\frac{2\mu^2}{\lambda}} e^{i\theta}$

Pick one w/ v real and positive: $\frac{v}{\sqrt{2}} = \sqrt{\frac{2\mu^2}{\lambda}}$ (Note v vs $v/\sqrt{2}$ convention)

- Linear realization: expand $\varphi = v + \frac{1}{\sqrt{2}}(\varphi_R + i\varphi_I)$

$$\Rightarrow V = \mu^2 \varphi_R^2 + \frac{\sqrt{\lambda \mu^2}}{2} \varphi_R (\varphi_R^2 + \varphi_I^2) + \frac{\lambda}{16} (\varphi_R^2 + \varphi_I^2)^2$$

$\Rightarrow$ Massless $\varphi_I \Rightarrow U(1)$ broken to nothing $\Rightarrow$ 1 Goldstone boson.

$\Rightarrow$ Massive φ_R w/ $m_{\varphi_R} = 2\mu$

- Non-linear realization: expand $\varphi = \frac{1}{\sqrt{2}}(v + \sigma(x)) \exp(i\, \pi(x)/F)$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \left(v + \frac{1}{\sqrt{2}}\sigma(x)\right)^2 \frac{1}{F^2} (\partial_\mu \pi)^2 - \left(-\frac{v^2}{4} + \mu^2 \sigma^2 + \frac{1}{2}\sqrt{\lambda}\,\mu \sigma^3 + \frac{1}{16}\lambda \sigma^4\right)$$

Setting $F = v \Rightarrow$ Canonical norm for π.

Related to linear version via field redef Now identify

$M_\pi = 0 \Rightarrow$ Goldstone & $M_\sigma = 2\mu$ is radial mode

Note: seeming paradox with $\varphi_I \varphi_I \to \varphi_I \varphi_I$, see H/W

Non-linear realization makes manifest that $\pi \to \pi + F\theta$ is sym

This is a "shift symmetry". All Goldstones have shift sym.

This is another signature of spontaneous sym breaking

Higgs mechanism (Abelian (ie $U(1)$))

Next, gauge the $U(1)$ symmetry:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + (\partial_\mu \varphi^* - ieA_\mu\varphi)(\partial_\mu\varphi + ieA_\mu\varphi) + \mu^2|\varphi|^2 - \frac{\lambda}{4}|\varphi|^4$$

minimize $V(\varphi) \Rightarrow |\langle\varphi\rangle| = \frac{v}{\sqrt{2}} = \sqrt{\frac{2\mu^2}{\lambda}}$

$$\Rightarrow \varphi(x) = \left(\frac{v+\sigma(x)}{\sqrt{2}}\right)\exp\left(i\pi(x)/v\right) \qquad (\text{set } F = v)$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \left(\frac{v+\sigma}{\sqrt{2}}\right)^2\left[-i\frac{\partial_\mu\pi}{v} + \frac{\partial_\mu\sigma}{v+\sigma} - ieA_\mu\right]\left[i\frac{\partial_\mu\pi}{v} + \frac{\partial_\mu\sigma}{v+\sigma} + ieA_\mu\right]$$
$$- \left(-\frac{\mu^4}{\lambda} + \mu^2\sigma^2 + \frac{1}{2}\sqrt{\lambda}\,\mu\sigma^3 + \frac{1}{16}\lambda\sigma^4\right)$$

Terms involving only A_μ: $\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}e^2v^2A_\mu^2 \Rightarrow m_A = ev$

Note: $m_\sigma = 2\mu$ and $m_\pi = 0$. σ is the "Higgs boson"

Simplify things by taking $m_\sigma \to \infty$ w/ v fixed $\Rightarrow$ σ decouples from low energy physics

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m_A^2\left(A_\mu + \frac{1}{m_A}\partial_\mu\pi\right)^2$$

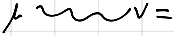
3.11

Performing gauge transform

$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$ and $\pi(x) \to \pi(x) - v\alpha$

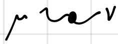
So can pick gauge where $\pi(x) = 0$ "unitary gauge"

More generally, recall that in Lorentz gauge ($\xi = 0$)



$= \frac{-i}{p^2 + i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$

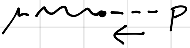
If we treat $\mathcal{L} \supset -\frac{1}{2}m_A^2 A_\mu^2$ as vertex



$= im_A^2 g^{\mu\nu}$

The π propagator as - - - $= \frac{i}{p^2}$ (massless Goldstone)

and cross-term



$= im_A(-ip_\mu)$

Then

$= \frac{-i}{p^2 + i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \underbrace{im_A^2 g^{\mu\nu} + m_A p^\mu \frac{i}{p^2} m_A p^\nu}_{= im_A^2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)} + \dots$

sum all m_A dep diags

$\overset{\downarrow}{=} \frac{-i}{p^2 - m_A^2 + i\varepsilon}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$

is massive prop in Lorentz gauge

This is how gauge boson eats Goldstone to become massive

Connection to superconductivity!

Non-Abelian generalization

Consider set of scalar fields φ_i that transform as

$$\varphi_i \rightarrow (1 + i\alpha^a T^a)_{ij}\varphi_j \quad \text{w/ } T^a \text{ generators of group } G$$

Let φ_i be real for simplicity (can split φ + φ^* into φ_R + φ_I if needed)

Note T^a_{ij} must be pure imaginary and since they are Hermitian $\Rightarrow$ anti symmetric.

Writing $t^a_{ij} = i T^a_{ij}$ $\Rightarrow$ t^a are real + anti-symmetric

$$\Rightarrow D_\mu \varphi = (\partial_\mu + g A^a_\mu t^a)\varphi$$

$$\Rightarrow \frac{1}{2}(D_\mu \varphi_i)^2 = \frac{1}{2}(\partial_\mu \varphi_i)^2 + g A^a_\mu(\partial_\mu \varphi_i \, t^a_{ij}\varphi_j) + \frac{1}{2}g^2 A^a_\mu A^{b\mu}(t^a\varphi)_i(t^b\varphi)_i$$

Next, assume $\langle \varphi_i \rangle = (\varphi_0)_i \neq 0$

$$\Rightarrow \Delta \mathcal{L} = \frac{1}{2} m^2_{ab} A^a_\mu A^{b\mu} \quad \text{w/ } m^2_{ab} = g^2 (t^a \varphi_0)_i (t^b \varphi_0)_i$$

Note any diagonal element in any basis has form

$$m^2_{aa} = g^2 (t^a \varphi_0)^2 \geq 0 \quad (\text{no sum})$$

$\Rightarrow$ All gauge bosons will recieve positive masses!

It is possible that some particular generator T^a of G leaves the vacuum invariant:

$$t^a \varphi_0 = 0$$

$\Rightarrow t^a$ will give no contribution to m^2_{ab} $\Rightarrow$ corresponding massless boson.

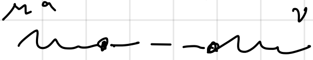
Next: note the gauge boson prop gets contribution from gauge - Goldstone mixing

$$\Delta \mathcal{L} = g A_\mu^a \partial_\mu \varphi_i (t^a \varphi_0)_i$$

This only involves components of φ_i that are parallel to the vector $t^a \varphi_0$

These vectors represent infinitesimal rotations of the vacuum $\Rightarrow$ the components of φ_i that contribute to $\Delta \mathcal{L}$ are the Goldstones.

Since they are massless, we can write



$$= \sum_j \left(g k^\mu (t^a \varphi_0)_j\right) \frac{i}{k^2} \left(-g k^\nu (t^b \varphi_0)_j\right)$$

gives m^2_{ab}

This plus mass term $\Rightarrow$

$$= i m^2_{ab} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$

which is a mass for the transverse modes!

Ex: SU(2) broken by doublet (complex) (4 dof)

$D_\mu \varphi = (\partial_\mu - i g A_\mu^a \tau^a)\varphi$ with $\tau^a = \sigma^a/2$

Let $\langle\varphi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}$

$\Rightarrow |D_\mu \varphi|^2 = \frac{1}{2} g^2 (0\ v)\, \tau^a \tau^b \begin{pmatrix}0\\v\end{pmatrix} A_\mu^a A^{b\mu} + \dots$

Using $\{\tau^a, \tau^b\} = \frac{1}{2}\delta^{ab}$ to symmetrize the matrix

$\Rightarrow \Delta \mathcal{L} = \frac{g^2 v^2}{8} A_\mu^a A^{a\mu}$ $\Rightarrow$ All three gauge bosons get mass.

One real scalar Higgs boson remains from φ

Now let φ transform as real triplet under SU(2)
(3 dof)

$\Rightarrow (D_\mu \varphi)^a = \partial_\mu \varphi^a + g \varepsilon^{abc} A_\mu^b \varphi^c$

Let $\langle\varphi\rangle = \begin{pmatrix}0\\0\\v\end{pmatrix}$ $\Rightarrow \mathcal{L} \supset \frac{1}{2}(D_\mu\varphi)^2 = \frac{g^2}{2}\left(\varepsilon^{abc} A_\mu^b (\varphi_0)^c\right)^2 + \dots$

$\Rightarrow \mathcal{L} \supset \frac{g^2}{2} v^2 \left(\varepsilon^{ab3} A_\mu^b\right)^2 = \frac{g^2}{2} v^2 \left((A_\mu^1)^2 + (A_\mu^2)^2\right)$

$\Rightarrow m_1 = m_2 = gv$ and $m_3 = 0$

$\Rightarrow$ Unbroken U(1)

Quantization of massive gauge bosons

* This relies on topics you will see in Advanced QFT

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2}m_A^2\left(A^a_\mu + \frac{1}{m_A}\partial_\mu \pi^a + \dots\right)^2 - \frac{1}{2\xi}\left(\partial_\mu A^a_\mu + \xi m_A \pi^a\right)^2$$

Then Faddeev - Popov gives

$$\det\left(\partial^\mu D_\mu + \xi m_A^2\right) = \int \mathcal{D}c\, \mathcal{D}\bar{c}\, \exp\left(-i\int d^4x\, \overbrace{\bar{c}^a\left(\partial^\mu D_\mu + \xi m_A^2\right)c^a}^{\text{new terms in } \mathcal{L}}\right)$$

Focus on kinetic terms

$$\mathcal{L}_{kin} = -\frac{1}{2}A^a_\mu\left(-g^{\mu\nu}\Box + \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu - m_A^2 g^{\mu\nu}\right)A^a_\mu - \frac{1}{2}\pi^a\left(\Box + \xi m_A^2\right)\pi^a - \bar{c}^a\left(\Box + \xi m_A^2\right)c^a$$

$$\nu, b \sim\!\sim\!\sim \mu, a = \frac{i}{p^2 - m_A^2}\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2 - \xi m_A^2}(1-\xi)\right)\delta^{ab}$$

Goldstone

$$b - - - - a = \frac{i}{p^2 - \xi m_A^2}\delta^{ab}$$

$$b \cdot\cdot\,\rangle\cdot\cdot\, a = \frac{i}{p^2 - \xi m_A^2}\delta^{ab}$$

and recall

$$\sum \varepsilon^\mu \varepsilon^{\nu *} = -\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m_A^2}\right)$$

Since physical processes must be ξ independent, clearly there must be delicate cancelations between diagrams.

Note the $\xi \to \infty$ is Unitary gauge where Goldstones and ghosts do not contribute ($\xi m^2 \to \infty$) and massive gauge boson prop is

$$i\Pi^{ab}_{A_\lambda A_\gamma}(p) = \frac{i\left(-g^{\mu\nu} + p^\mu p^\nu / p^2\right)}{p^2 - m_A^2} \delta^{ab}$$

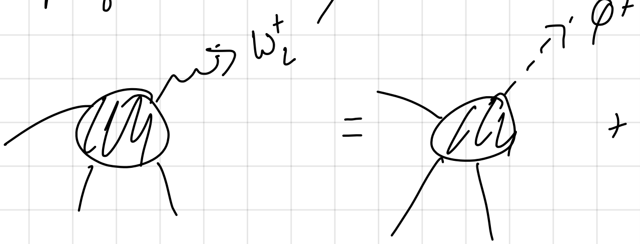
Goldstone Boson Equivalence Thm

Massive gauge boson at rest, no distinction between polarizations.

Massive gauge boson w/ momentum >> mass $\Rightarrow$ distinction between transverse and longitudinal polarizations.

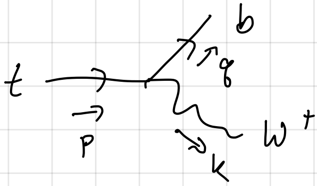
Longitudinal polarization retains Goldstone nature

See how this work through example of top quark decay:



$+\,\mathcal{O}\left(\frac{m_W^2}{E^2}\right)$

Top decay



Naively guess $\Gamma \sim \frac{g^2}{4\pi} m_t$

Actual expression has $(m_t/m_W)^2$ enhancement from Goldstone derivative coupling

First, calculate full answer:

W only couples to left handed fermions.

$$iM = \frac{ig}{\sqrt{2}}\, \bar{u}(q)\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)u(p)\,\varepsilon^*_\mu(k)$$

$$\Rightarrow \frac{1}{2}\sum_{\text{spins}}|M|^2 = \frac{g^2}{2}\left[q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu} q\cdot p\right]\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}\right]$$

$$= \frac{g^2}{2}\left[q\cdot p + 2\,\frac{(k\cdot q)(k\cdot p)}{m_W^2}\right]$$

Take $m_b = 0 \Rightarrow 2q\cdot p = 2q\cdot k = m_t^2 - m_W^2$

$$2k\cdot p = m_t^2 + m_W^2$$

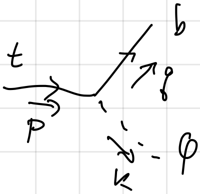
$$\Rightarrow \Gamma = \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+2\frac{m_W^2}{m_t^2}\right)$$

Next, we can derive this enhancement by considering

$$\Delta\mathcal{L} = -\lambda_t \varepsilon^{ab}\bar{Q}_{La}\varphi^\dagger_b t_R + h.c. \quad \text{(from Higgs coupling)}$$

$$\Rightarrow \frac{\lambda_t^2}{g^2} = \frac{m_t^2}{2m_W^2}$$

Write $\varphi^\pm = \frac{1}{\sqrt{2}}(\varphi_R \pm i\varphi_I) \Rightarrow \Delta\mathcal{L} = \lambda_t \bar{b}_L \varphi^+ t_R$

$\Rightarrow$ 

$$= i\mathcal{M} = i\lambda_t \bar{u}(q)\left(\frac{1+\gamma_5}{2}\right)u(p)$$

neglecting mass of φ^+ $\Rightarrow \frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2 = \lambda_t^2\, p\cdot q \Rightarrow \Gamma = \frac{\lambda_t^2}{32\pi}m_t = \frac{g^2}{64\pi}\frac{m_t^3}{m_W^2}$

Note that we can get full expression by only including transverse W's in spin sum

$\sum \varepsilon_\mu \varepsilon_\nu^* = -g_{\mu\nu}$ and including m_W mass for φ^+ in kinematics reproduces full result.

Reproduces physics of $\varepsilon_L^\mu = \frac{k^\mu}{m} + \mathcal{O}(m/E_k)$